Renormalization

We know that after using all δ-functions: $M \sim \delta^4(p_{tot\,in} - p_{tot\,out}) \int f(p_i, q_k) \prod_k \frac{d^4 q_k}{(2\pi)^4}$ (leftover q after using δ-functions)

In many scenarios the remaining ∫ diverges. This is usually at large q_k.

We can render the ∫ finite by "regularizing" it, e.g. introduce factors $\frac{-\tilde{M}^2 c^2}{q_k^2 - \tilde{M}^2 c^2}$ } Pauli-Villars massive regulator

where $\tilde{M}$ is a free parameter (not related to m_k). ↑ increases power of q_K in denominator

In the end we take $\tilde{M} \to \infty$ so factor → 1. If we are lucky then the resulting form of the integral will split into two parts: One will depend on $\tilde{M}$ and → ∞ (as $\tilde{M} \to \infty$).
One will be ind. of $\tilde{M}$ and remain finite.

If we're really lucky the $\tilde{M}$-dep. parts will look like: $m_p = m_b + \delta m(\tilde{M})$
$g_p = g_b + \delta g(\tilde{M})$

In this case we can say: m_b and g_b are the inputs in our original theory. We naively expected these to be the finite values measured in experiments. That was dumb! m_b and g_b are the true "bare" values that should fundamentally define the theory, but what we measure are the "physical" values m_p and g_p.

We know m_p and g_p are finite, but m_b and g_b could be anything, in particular divergent enough to cancel δm and δg!

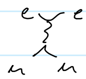
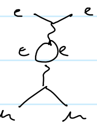
If this story plays out, then we call the theory renormalizable.

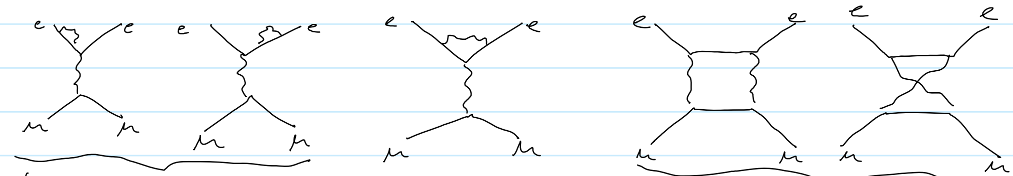
Note: You can always regularize the theory to make it finite. The key to renormalizability is removing the regulator in a consistent way.

Gerard t'Hooft demonstrated the full renormalizability of the Standard Model.

Unfortunately, if we try to consider perturbative quantum gravity, we find that it is a non-renormalizable theory.

What are we to make of this? We will soon see.

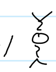
We considered  +  but there are other 4th order (1-loop) diagrams. Each play certain roles as described:

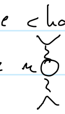


electron self-energy corrections — renormalizes electron mass

vertex correction — renormalizes electron magnetic moment

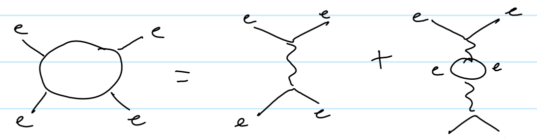
These are finite and so do not contribute to renormalization

These 3 separately contribute to the electron charge renormalization (along w/ ) but the combined effect of these three cancels, so only  plays a nontrivial role.

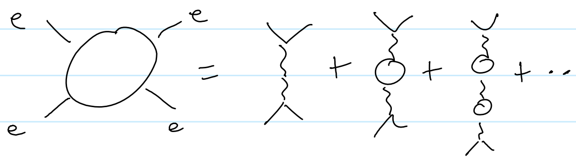
This is a good thing since the corrections to the charge from these three diagrams would be mass dependent leading to different effective charges for e, μ, τ. But we observe the same effective charges which reflects that their contributions cancel. Note:  is mass independent. We could include , but this would renormalize the e, μ, τ charges the same.

Renormalization in QCD

In QED we found that vacuum polarization led to effective charge screening which made the electric charge "run" to larger values at smaller distances (or at larger $|q^2|$ momentum transfer).

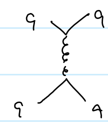
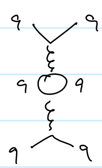
That is:  $\Rightarrow \alpha(|q^2|) = \alpha(0)\left[1 + \frac{\alpha(0)}{3\pi}\ln\left(\frac{|q^2|}{m_e^2c^2}\right)\right]$ for $|q^2| >> m_e^2c^2$

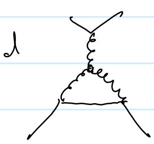
We can actually sum over all contributions of this form since the series ends up being geometric:

 $\Rightarrow \alpha(|q^2|) = \dfrac{\alpha(0)}{1 - \frac{\alpha(0)}{3\pi}\ln\left(\frac{|q^2|}{m_e^2c^2}\right)}$ $\quad |q^2| >> m_e^2c^2$

Note: This does not count all possibilities, e.g. , but these are the leading terms at each order.

Now $\alpha(|q^2|)$ increases w/ increasing $|q^2|$ and even blows up when $\ln\left(\frac{|q^2|}{m_e^2c^2}\right) = \frac{3\pi}{\alpha(0)} = \frac{3\pi}{e}\sqrt{\frac{\hbar c}{4\pi}}$ but this is at 10^{280} MeV so not too troubling.

In QCD we have  corrected by  which also leads to screening.

but additionally we have  and  and  which leads to anti-screening.

Now which effect wins depends on the number of different quark flavors (screening) f, and the number of colors n which impacts both the number of quarks (screening) and gluons (anti-screening).

Evaluating loop diagrams in QCD require FP ghosts and is beyond us, but the result analogous to the QED case is:

$$\alpha_s(|q^2|) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(11n - 2f)\ln\left(\frac{|q^2|}{\mu^2}\right)}$$

$|q^2| >> \mu^2$ ← this is some scale at which we set a reference value of α_s (similar to $\alpha_e(0)$ in QED, but we can't use $\alpha_s(0)$ since it blows up).

Now if $11n - 2f \begin{cases} > 0 & \text{anti-screening wins} \\ < 0 & \text{screening wins (like QED)} \end{cases}$

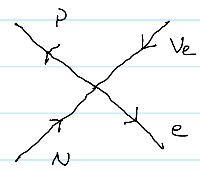
In the SM $n=3$, $f=6 \Rightarrow 33-12>0$ so anti-screening wins and $\alpha_s(|q^2|)$ decreases w/ increasing $|q^2|$. This leads to asymptotic freedom which allows us to effectively use perturbation theory in QCD at short distances, e.g. inside mesons and baryons!

Of course one could try to extrapolate from this that at low $|q^2|$ or large distances the $\alpha_s(|q^2|) \to \infty$ and hence explain confinement, but in this regime perturbation theory breaks down.

As a side note, theorists like to play around with "pure" QCD (only gluons) since it is actually a finite theory (no renormalization needed).

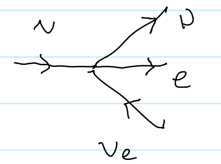
Non-renormalizability and Effective Field Theories

An insightful historical anecdote:
Prior to our complete understanding of both QCD and the weak interactions, Fermi had proposed a field theory model to describe nuclear beta decay. His theory included four fermion fields (p, n, e, ν_e) and an interaction vertex



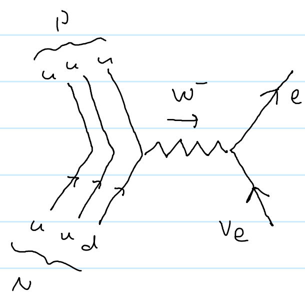
Fermi "four-point" interaction.

This could obviously describe beta decay $N \to p + e + \bar{\nu}_e$ via



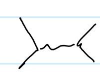
Aside from experimental evidence to suggest a more fundamental picture, what is really interesting is that the theory itself predicted its own breakdown. The theory based on the four-point interaction was non-renormalizable.

Now remember, non-renormalizability is only a sickness if certain quantities are taken to infinity. However, in this case we now understand why this theory is broken. In truth the underlying process is:



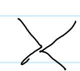
In a sense the Fermi four-point theory is collapsing the W^- line to zero. To see the conditions for when this reasonable we can just consider the amplitude M:

$$[\bar{u}(N)\gamma^\mu(1-\gamma^5)u(p)]\ \frac{-ig_{\mu\nu} - q_\mu q_\nu/M_W^2c^2}{q^2 - M_W^2c^2}\ [\bar{v}(\nu_e)\gamma^\nu(1-\gamma^5)u(e)] \qquad VPV$$



for $q \ll M_W c$

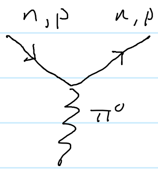
$$[\bar{u}(N)\gamma^\mu(1-\gamma^5)u(p)]\ \frac{ig_{\mu\nu}}{M_W^2c^2}\ [\bar{v}(\nu_e)\gamma^\nu(1-\gamma^5)u(e)] \qquad \sim VV$$

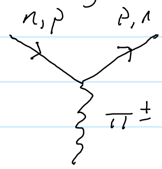


So we see now that for $q \ll M_W c$, the four-point interaction is useful, but when $q \gtrsim M_W c$, we need to use the full underlying theory.

Other Effective Field Theories

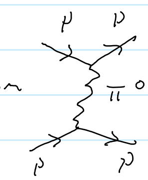
Prior to our understanding of QCD w/ quarks and gluons, there was a very successful model of the strong interactions in terms of matter fields for the proton and neutron and gauge fields corresponding to the pions w/ fundamental vertices:



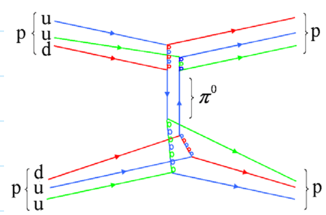


In fact this was an SU(2) gauge theory acting on nucleon doublets $\binom{n}{p}$ w/ a gauge field $\pi^0, \pi^\pm$ for each generator of SU(2).

As you can imagine, calculating processes w/ this theory, e.g. $p+p \to p+p$ w/ $M_{leading}$ from



is much easier that working w/ the true fundamental theory w/ $M_{leading}$ from



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Wilson's Approach to Renormalization

The example of Fermi's model of β-decay provides a natural and universal idea that was formalized by Ken Wilson in the 70's.

The basic idea is that a non-renormalizable theory (with unremovable ∞'s) is really an effective theory that should only be used up to some energy scale. Beyond that, we should instead work with the more fundamental "ultra-violet completion" of the theory. The expectation is that the UV-completion itself would either be renormalizable or even finite.

Turning this around, Wilson introduced the notion of moving from fundamental descriptions to effective theories by "integrating out" the higher energy degrees of freedom, and working only in terms of the lower energy degrees of freedom. Note, this is exactly what we do to get to the Fermi model, i.e. ignore the highest energy (most massive) part of the fundamental description.

BTW, this program does not have to connect field theories to field theories. For example we could approximate a discrete atomic system at large distances by an approximate continuum description. This effective field theory description would be non-renormalizable and would break down at energy scales associated w/ the inter-atomic spacing.

Quantum Gravity

As we noted earlier, perturbative quantum gravity is non-renormalizable. Our interpretation now is that it is still a useful effective field theory that should be replaced at some appropriate energy scale by its UV-completion.

What scale? If we take the fundamental constants $\hbar, c$ and the gravitational constant G, then from these we can form a fundamental (gravitational) mass or energy scale:

$$M_{planck} = \sqrt{\frac{\hbar c}{G}} \approx 10^{-8} kg \quad \text{compare w/ } M_{weak} \approx M_{Higgs} \approx 10^{-25} kg$$
$$E_{planck} = M_{planck} c^2 \approx 10^{19} \text{ GeV}$$
$$l_{planck} = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} m$$

Clearly we can rely on perturbative quantum gravity up to very high energies / short distances. However, if we ever want an accurate description of gravity very near the Big Bang or the singularity of a BH, we will need a UV-completion.

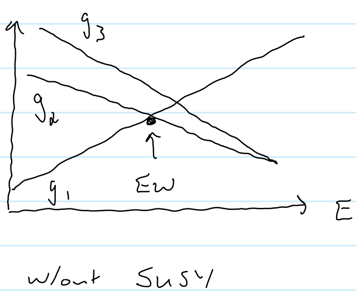
What UV-completes QG?

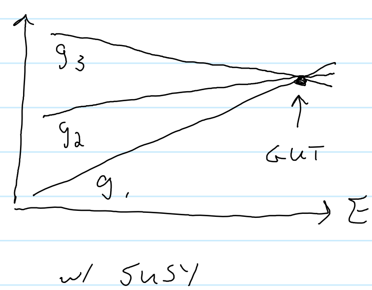
One contender is based on a straightforward quantization of GR and goes by the name of Loop Quantum Gravity. LQG has some success, but really only serves to address the problem of QG itself.

String Theory on the other hand, handles QG and much more. Its fundamental assertion is that points → strings (NOT an attempt to quantize gravity). The benefits though are numerous:

i) We get a consistent (actually finite) quantum theory of gravity. The theory actually has a natural built in regulator (the string length scale).

ii) String theory naturally comes equipped w/ supersymmetry which is actually a crucial component in the story of gauge force unification:





iii) String theory comes equipped w/ gauge symmetries $SO(32)$, $E8 \times E8$ (496 generators) which are plenty big enough to accommodate a unification of $SU(3) \times SU(2)_L \times U(1)_Y$ (12 generators).

iv) String theory is awesome!!